

ON THE REGULARITY OF VARIETIES HAVING AN EXTREMAL SECANT LINE

MARIE-AMÉLIE BERTIN

ABSTRACT. The Castelnuovo-Mumford regularity of a non degenerate variety of degree d and dimension n of \mathbb{P}^r is conjectured to be at most $d - r + n + 1$ [7]. This conjecture is known to hold only in a few cases: curves [10], smooth surfaces ([16],[11]), and smooth varieties of codimension 2 [1]. Varieties with an extremal secant line are $(d - r + n)$ -irregular and the classification of smooth $(d - r + 1)$ -irregular curves in [10] shows that they all possess an extremal secant line but the elliptic normal curve and some smooth rational curves in \mathbb{P}^{d-1} . We give here a complete classification of varieties having an extremal secant line and show that their regularity is $d - r + n + 1$.

1. INTRODUCTION

Let X be a complex n -dimensional variety, non degenerate and of degree d in \mathbb{P}^r . The k -regularity of X has been defined by D.Mumford [14] by the following vanishing

$$H^i(\mathbb{P}^r, \mathcal{I}_{X|\mathbb{P}^r}(k - i)) = 0 \quad \text{for all } i \geq 1.$$

The k -regularity condition implies the $(k + 1)$ -regularity condition, so that one defines the *Castelnuovo-Mumford regularity* of X , as the the least integer k such that X is k -regular. We denote by $\text{reg}(X)$ the regularity of X . This obscure vanishing condition find its origin in G.Castelnuovo's work on the postulation of a variety X [6]; for the regularity of X gives an upper bound on the least integer k for which hypersurfaces of degree $s \geq k$ cut on X a complete regular linear system. The regularity X also encodes pieces of information on the syzygies of the equations defining X . Indeed, let I_X be the saturated ideal of X in $R = \mathbb{C}[x_0, \dots, x_n]$ and

$$0 \rightarrow L_{r+1} \rightarrow \cdots \rightarrow L_0 \rightarrow R \rightarrow R/I_X \rightarrow 0$$

be a minimal free graded resolution of the R -module R/I_X ; the variety X is k -regular if and only if for all $i \geq 0$, one can find a base of L_i , which elements are at most of degree $k + i$ [2], definition 3.2.

This algebraic characterization of k -regularity has an elementary geometric consequence: the existence of a k -secant line to X implies that $\text{reg}(X) \geq k$. One can deduce from Bertini's theorem on linear systems that X cannot have any k -secant line for $k > d - r + n + 1$; we will thus call *extremal* a $(d - r + n + 1)$ -secant line to X . Varieties for which there exist extremal secant lines are therefore $(d - r + n)$ -irregular. The *regularity conjecture* ([7],[8]) foresees that $\text{reg}(X) \leq d - r + n + 1$. It has so far only been proved for curves [10], smooth surfaces ([16], [11]) and smooth 2-codimensional varieties [1]. Extremal varieties for the conjecture have only been classified for curves by Gruson, Lazarsfeld and Peskine in [10]. They are smooth

Key words and phrases. Castelnuovo-Mumford regularity, scrolls.

rational curves with an extremal secant line except for the elliptic normal curve and some rational curves in \mathbb{P}^{d-1} .

Varieties having an extremal secant line thus form a good sample for testing the regularity conjecture.

Conventions and notations: We will use Hartshorne's book conventions on projective bundles. Consider n positive integers $1 \leq a_1 \leq \dots \leq a_n$, the smooth rational normal scroll $S(a_1, \dots, a_n)$ is the tautological embedding of the projective bundle $\mathbb{P}(\oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i))$ in $\mathbb{P}^{n-1 + \sum_{i=1}^n a_i}$; a non degenerate cone of vertex \mathbb{P}^k over $S(a_1, \dots, a_n)$ in $\mathbb{P}^{n+k-\sum_{i=1}^n a_i}$ will be denoted by $S(\underbrace{0, \dots, 0}_{k\text{-times}}, a_1, \dots, a_n)$.

Here we will prove the following results.

First of all we show the existence of non degenerate n -folds X of degree d in \mathbb{P}^r , having an extremal secant line, for any $d > r - n + 1$, n and $r \geq 2n + 1$.

Theorem 1. *Let us fix an integer n , then choose n non zero positive integers $a_1 \leq \dots \leq a_n$ and a positive integer $d > \sum_{i=1}^n a_i$. Let us pick also two non zero positive integers k_1 and k_2 such that $k_1 + k_2 = d - \sum_{i=1}^n a_i$ and an injection of $\mathcal{O}_{\mathbb{P}^1}$ -module*

$$\mathcal{O}_{\mathbb{P}^1}(-k_1) \oplus \mathcal{O}_{\mathbb{P}^1}(-k_2) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^1}^2 \oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i).$$

The cokernel map

$$\mathcal{O}_{\mathbb{P}^1}^2 \oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i) \xrightarrow{\beta} \oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(b_i)$$

gives a morphism of rational scrolls from $\overline{X} = S(b_1, \dots, b_n)$ to $S = S(0, 0, a_1, \dots, a_n)$ restriction of a linear projection between their linear spans that induces an isomorphism of rational scrolls between \overline{X} and its image X . The vertex of the cone S is an extremal secant line for the n -fold scroll X . Moreover any rational n -fold scroll having an extremal secant line can be realized this way.

This is completed by our classification theorem.

Theorem 2. *Let X be a non degenerate variety in \mathbb{P}^r with $r - n \geq 2$ of degree d and dimension $n \geq 2$. Assume that X has an extremal secant line l . Then X is either*

- a cone over the Veronese surface V in \mathbb{P}^5 or
- a cone over a smooth rational scroll Σ , such that l is an extremal secant line for Σ or
- a cone over the Veronese surface V' in \mathbb{P}^4 .

It is known that $\text{reg}(V) = 2$ and $\text{reg}(V') = 3$. From the algebraic characterization of k -regularity, if X is a cone over Σ we have $\text{reg}(X) = \text{reg}(\Sigma)$, so that to show that varieties with an extremal secant line satisfy the regularity conjecture we only need to show it for n -fold rational scrolls.

Theorem 3. *Let X be a n -fold scroll over a smooth curve C of genus g , of degree d in \mathbb{P}^r . Then we have the inequality*

$$\text{reg}(X) \leq d - r + n + 1.$$

Acknowledgments This work was made while I was a Ph.D. student at Columbia University. Most of it constitutes my Ph.D. dissertation [4] and grew out of discussions with my advisor Prof. Henry C. Pinkham, to whom I express my gratitude. The last theorem was inspired by personal notes of Prof. Lazarsfeld [12], kindly communicated to us by their author.

2. RATIONAL SCROLLS WITH AN EXTREMAL SECANT LINE

Beside the case where bisecant lines are extremal secant lines for X , i.e. X is a variety of minimal degree, the existence of varieties with extremal secant line is not clear. The latest construction of such varieties is due to Lazarsfeld. He constructs in Remark 1 of [11] smooth surfaces of $\mathbb{P}^r, r \geq 5$ with an extremal secant line as 1-codimensional subvarieties of a rational normal 3-fold scroll. The construction readily generalizes to higher dimension and furnishes smooth varieties (actually rational scrolls) of any degree $d > r - n + 1$ in $\mathbb{P}^r, r \geq 2n + 1$ having an extremal secant line.

We explain here how to construct any rational n -fold scroll X that has an extremal secant line. First we note that X must be a 2-codimensional subvariety of a singular rational normal scroll.

Proposition 1. *Let X be a variety in \mathbb{P}^r having an extremal secant line l . The image of X by the linear projection π_l of center l is a n -dimensional variety X' of minimal degree in \mathbb{P}^{r-2} . Moreover if X is a smooth rational scroll so is X' .*

Proof. If we project X from this line l to \mathbb{P}^{r-2} , we get a variety X' of degree d' and dimension $n' \leq n$ in \mathbb{P}^{r-2} . Since X is non degenerate, $n = n'$. It follows that $d' = r - n - 1$, hence X' is of minimal degree.

The rational scroll X is the image of a smooth rational normal scroll Z of degree d and dimension n , by a linear projection. We deduce from the classification of varieties of minimal degree ([5],[9]) that X' is a rational normal scroll. The induced rational map g from Z to $X' = \pi_l(X)$ is thus an elementary transformation (cf.[13]) of Z along $Y = \{p_1, \dots, p_s\}$. It follows that X' is the tautological embedding of a projective bundle, elementary transform of $\mathbb{P}(\mathcal{E})$; therefore X' is smooth. \square

Thus our n -fold scroll lies on the cone $\langle l, X' \rangle = S(0, 0, a_1, \dots, a_n)$ for some integers $1 \leq a_1 \leq \dots \leq a_n$. We can reverse the process; on any rational normal scroll $S = S(0, 0, a_1, \dots, a_n)$, we can construct a smooth codimension 2 subscroll X having the vertex of the cone S as extremal secant line.

Theorem 1. *Fix an integer n , then choose n non zero positive integers $a_1 \leq \dots \leq a_n$ and a positive integer $d > \sum_{i=1}^n a_i$. Let us pick also two non zero positive integers k_1 and k_2 such that $k_1 + k_2 = d - \sum_{i=1}^n a_i$ and an injection of $\mathcal{O}_{\mathbb{P}^1}$ -module*

$$\mathcal{O}_{\mathbb{P}^1}(-k_1) \oplus \mathcal{O}_{\mathbb{P}^1}(-k_2) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^1}^2 \oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i).$$

The cokernel map

$$\mathcal{O}_{\mathbb{P}^1}^2 \oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i) \xrightarrow{\beta} \oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(b_i)$$

gives a morphism of rational scrolls from $\overline{X} = S(b_1, \dots, b_n)$ to $S = S(0, 0, a_1, \dots, a_n)$ restriction of a linear projection between their linear spans that induces an isomorphism of rational scrolls between \overline{X} and its image X . The vertex of the cone S is an extremal secant line for the n -fold scroll X . Moreover any rational n -fold scroll having an extremal secant line can be realized this way.

Proof. Let us denote by \tilde{S} the projective bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^2 \oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i))$ in $\mathbb{P}^1 \times \mathbb{P}^r$, where $r = n + 1 + \sum_{i=1}^n a_i$. The second projection π_2 from the product $\mathbb{P}^1 \times \mathbb{P}^r$ embeds \tilde{S} in \mathbb{P}^r as a cone S of vertex l over the smooth rational normal scroll $X' = S(a_1, \dots, a_n)$. Its restriction to \tilde{S} is a desingularization of S of exceptional

locus $E = \mathbb{P}^1 \times l = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^2)$. A simple Chern class computation shows that $\sum b_i = d + n - 1$, with $b_i \geq 1$ for all $1 \leq i \leq n$. Let us denote by \overline{X} the smooth rational normal scroll $S(b_1, \dots, b_n)$ in \mathbb{P}^{d+n-1} . Since $h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-k_1) \oplus \mathcal{O}_{\mathbb{P}^1}(-k_2)) = 0$, the map α defines a morphism f from \overline{X} to S , which image X is a non degenerated rational scroll of S in \mathbb{P}^r . Moreover since \overline{X} is linearly normal and S non degenerated, f is the restriction to \overline{X} of a linear projection. Since the restriction β_2 of β to $\sum_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i)$ has to be an injection of $\mathcal{O}_{\mathbb{P}^1}$ -modules, it defines an elementary transformation between $S(a_1, \dots, a_n)$ and \overline{X} , induced by a linear projection from \mathbb{P}^{d+n-1} to \mathbb{P}^{r-2} , which factors through f and the projection of \mathbb{P}^r from the line l . Therefore X is smooth of dimension n , so that f induces an isomorphism of rational scrolls between \overline{X} and X and $\deg X = d$.

We denote by \tilde{X} the strict transform of X in \tilde{S} . Let σ denote the embedding map $\mathbb{P}^1 \rightarrow l \subset \mathbb{P}^r$. Notice that the diagonal $\Delta = \{(t, p) \in E : p = \sigma(t)\}$ in E is isomorphic to l by π_2 , so that the schemes $Z = \tilde{X} \cap \Delta$ and $X \cap l$ are isomorphic by π_2 . For each $i = 1, 2$, the injections induced by α , $\mathcal{O}_{\mathbb{P}^1}(-k_1) \xrightarrow{\alpha_{1,i}} \mathcal{O}_{\mathbb{P}^1}^2$, define in E over \mathbb{P}^1 a rational curve $C_i = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(k_i))$. Let $\oplus \mathbb{C}^{n_i} w_i$ denote the cokernel of

$$\mathcal{O}_{\mathbb{P}^1}(-k_1) \oplus \mathcal{O}_{\mathbb{P}^1}(-k_2) \xrightarrow{\alpha_1 = \alpha_{1,1} + \alpha_{1,2}} \mathcal{O}_{\mathbb{P}^1}^2;$$

Remark that $1 \leq n_i \leq 2$. Since for each $i = 1, \dots, s$, the line $\pi_1^{-1}(w_i) \cap E$ meets Δ , the scheme Z is supported on $\{(w_1, \sigma(w_1)), \dots, (w_s, \sigma(w_s))\}$ in Δ . If $n_i = 2$, the smoothness of each curve C_i at each point of $C_i \cap \Delta$, shows that $(w_i, \sigma(w_i))$ is on $C_i \cap \Delta$ for both curve C_1 and C_2 . The cotangent space of $\tilde{X} \cap E$ at $(w_i, \sigma(w_i))$ is then the direct sum of the cotangent spaces of C_i at this point. Therefore the multiplicity of $\tilde{X} \cap \Delta$ at $(w_i, \sigma(w_i))$ is 2. From this, we deduce that the length of Z is $\sum n_i = d - r + n + 1$. \square

3. THE CLASSIFICATION OF VARIETIES HAVING AN EXTREMAL SECANT LINE

Let X be a n -dimensional variety in \mathbb{P}^r having an extremal secant line l . From Gruson, Lazarsfeld and Peskin's classification of $(d - r + 1)$ -irregular curves [10], if $n = 1, r \geq 3$, X is a *smooth rational curve*. This and Del Pezzo-Bertini's theorem [5],[9] that classifies varieties of minimal degree, are the key ingredients for our classification theorem.

Theorem 2. *Let X be a non degenerate variety in \mathbb{P}^r with $r - n \geq 2$ of degree d and dimension $n \geq 2$. Assume that X has an extremal secant line l . Then X is either*

- a cone over the Veronese surface V in \mathbb{P}^5 or
- a cone over a smooth rational scroll Σ , such that l lies is an extremal secant line for Σ or
- a cone over the Veronese surface V' in \mathbb{P}^4 .

Proof. Let $\tilde{X} \xrightarrow{\phi} X$ be a desingularization of X of exceptional locus E . We want to determine the linearly normal variety \overline{X} of which X is projection. We show by induction on n that \overline{X} is a variety of minimal degree d . To do so, we need the existence of an hyperplane section of X containing $l \cap X = \{p_1, \dots, p_s\}$ that is desingularized by ϕ .

Lemma 2.1.

1. *The variety X is smooth in a neighborhood of $X \cap l$.*
2. *The variety \tilde{X} satisfies $h^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$.*

Proof of (1). This is true for $n = 1$ since then X is a smooth rational curve [10]. Let us assume it is true for $k < n$.

Consider the linear system \mathcal{L} of hyperplanes H containing l . The generic element of the restriction to X of \mathcal{L} satisfies our induction hypothesis so that any p_i in $X \cap l$ is a smooth point of $h = X \cap H$ for H generic. Since any smooth point p on a Cartier divisor D of a variety X is a smooth point of X , p_i is a smooth point of X . \square

Proof of (2). Since $h^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$ is a birational invariant for smooth varieties, we deduce that

$$h^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = h^1(\tilde{X}', \mathcal{O}_{\tilde{X}'})$$

for any desingularization \tilde{X}' of the variety X' . Moreover since X' is a variety of minimal degree, hence a cone over a smooth rational variety, we have $h^1(\tilde{X}', \mathcal{O}_{\tilde{X}'}) = 0$. \square

Let us now prove that X is the image by a regular projection of a variety of minimal degree \overline{X} .

Lemma 2.2. *Let H be a generic hyperplane section of X . Recall that $\phi: \tilde{X} \rightarrow X$ is a desingularization of X .*

1. *A generic element of the linear system $|\phi^*(\mathcal{O}_X(H))|$ is smooth and the dimension of this system is $d + n - 1$.*
2. *The rational map $f: \tilde{X} \dashrightarrow \mathbb{P}^{d+n-1}$ defined by this system is in fact regular and maps \tilde{X} onto a variety \overline{X} of dimension n and degree d .*

Proof. For $n = 1$, [10], the curve X is a smooth rational curve of degree d , hence is a regular projection of some rational normal curve in \mathbb{P}^d .

For $1 \leq k < n$ assume that for any k -dimensional varieties X having an extremal secant line, the total transform $|\mathcal{O}_{\tilde{X}}(H)|$ of the linear system of hyperplane sections of X , gives a map from \tilde{X} to \mathbb{P}^{d+k-1} which image \overline{X} is a k -dimensional variety of degree d , that projects onto X .

Let us consider on \tilde{X} the total transform \mathcal{L} of the system cut out on X by the system of hyperplanes containing l . Let D be a generic hyperplane section of X containing $X \cap l$; a generic element of D is irreducible and ϕ^*D of \mathcal{L} is smooth.

Indeed ϕ^*D is smooth away from its base locus by Bertini's theorem on singularities of a generic member of a linear system. The variety X is smooth at p_1, \dots, p_s , hence the base locus of \mathcal{L} consists of the s points $\phi^{-1}p_1, \dots, \phi^{-1}p_s$. The divisor D is an irreducible hypersurface of X by Bertini's irreducibility theorem ($r - 2 \geq 2$); it has an extremal secant line by construction, hence D is smooth in a neighborhood of p_1, \dots, p_s by lemma 2.1 (1). The Cartier divisor ϕ^*D is then smooth at $\phi^{-1}p_1, \dots, \phi^{-1}p_s$. This also implies that a generic element of $|\mathcal{O}_{\tilde{X}}(\phi^*H)|$ is smooth.

We can thus find an irreducible and non degenerated hyperplane section of X , H_0 such that ϕ^*H_0 is a smooth element of \mathcal{L} . We have the following exact sequence:

$$0 \rightarrow \mathcal{O}_{\tilde{X}} \rightarrow \mathcal{O}_{\tilde{X}}(\phi^*H) \rightarrow \mathcal{O}_{\phi^*H_0}((\phi^*H)|_{\phi^*H_0}) \rightarrow 0.$$

By lemma 2.1 (2), $h^0(\mathcal{O}_{\tilde{X}}(\phi^*H)) = 1 + h^0(\mathcal{O}_{\phi^*H_0}((\phi^*H)|_{\phi^*H_0}))$. The restriction of ϕ to the closure of $\phi^*H_0 \setminus E$ in \tilde{X} is a desingularization of H_0 so that we can apply the induction hypothesis to a generic hyperplane section h of H_0 to get:

$$h^0(\tilde{H}_0, \mathcal{O}_{\tilde{H}_0}(\phi|_{\tilde{H}_0}^*h)) = d + n - 1.$$

Since a generic element of $|\mathcal{O}_{\phi^*H_0}(\phi^*H)|_{\phi^*H_0}|$ doesn't meet E ,

$$h^0(\phi^*H_0, \mathcal{O}_{\phi^*H_0}(\phi^*H)|_{\phi^*H_0}) = h^0(\tilde{H}_0, \mathcal{O}_{\tilde{H}_0}(\phi^*H)|_{\tilde{H}_0}) = d + n - 1.$$

Since $|\mathcal{O}_{\tilde{X}}(\phi^*H)|$ is base point free, the degree of \overline{X} is d and the rational map ψ that $|\mathcal{O}_{\tilde{X}}(\phi^*H)|$ defines is regular.

The dimension of \overline{X} is at least $n - 1$. Since X is the union of the points in the pencil formed by 2 independant hyperplane sections H_1 and H_2 of X containing $l \cap X$, \overline{X} is cut out by the pencil generated by the images of the strict transforms of H_1 and H_2 in \tilde{X} and therefore has dimension n .

The linear system $|\mathcal{O}_{\tilde{X}}(\phi^*H)|$ generically separates points, so that the map ψ is birational onto its image. Its inverse ψ^{-1} is then a rational map from \overline{X} onto X that extends to a linear projection π from \mathbb{P}^{d+n-1} onto \mathbb{P}^r , since \overline{X} is linearly normal. The map ψ^{-1} is regular since \overline{X} and X have the same dimension and the same degree. \square

We can now use the Del Pezzo-Bertini theorem to conclude. Assume that $r - n = 2$, then \overline{X} is a cone over the Veronese surface V in \mathbb{P}^5 and X is the linear projection of \overline{X} by a point. Since the extremal secant line l meets X at smooth points, it must be the image of a 3-secant 2-plane of V , hence the center of projection lies in the linear span of V and X is a cone over the Veronese surface in \mathbb{P}^4 . If $r \geq n + 3$ the variety \overline{X} can be neither a cone over the Veronese surface nor the Veronese surface itself, unless $\overline{X} = X$. Indeed, if it were so, X would be a cone $\langle L^k, V' \rangle$ over the generic projection of the Veronese to \mathbb{P}^4 or over the Steiner surface in \mathbb{P}^3 , with $k = n - 3$. So clearly, we would have $r \leq n + 2$. If $n = 2$, we cannot have any projection of the Veronese surface either.

Hence \overline{X} must be a rational scroll unless X was already a variety of minimal degree. Since the projection π is regular, if \overline{X} is smooth, X is a smooth rational scroll. If \overline{X} is a cone $\langle L^k, S^{n-k-1} \rangle$ over a smooth scroll of minimal degree S^{n-k-1} , X must be the cone $\langle L^k, \pi(S^{n-k-1}) \rangle$, hence is a cone over a smooth rational scroll. Any extremal secant line moreover has to lie in the linear space generated by S^{n-k-1} , for it has to meet X at smooth points. \square

4. THE REGULARITY OF SMOOTH SCROLLS

In this section we prove that varieties having an extremal secant line are $d - r + n + 1$ -regular. By the classification theorem, we only need to prove the regularity of smooth scrolls having an extremal secant line.

This will follow from this more general result on the regularity of smooth scrolls over a smooth curve C .

Theorem 3. *Let X be a smooth n -dimensional scroll over a smooth curve C of genus g , embedded in \mathbb{P}^r as a non degenerate variety of degree d . The regularity of X is bounded by $d - r + n + 1$, as predicted by the regularity conjecture.*

The proof is closely related to Gruson, Lazarsfeld and Peskine's proof of the regularity conjecture for curves. As they do, we first describe X as a projection

from a product of a variety \mathcal{S} that is scheme theoretically a vanishing locus by a Beilinson type construction [15]; in our case \mathcal{S} is the the ruled subvariety of $C \times \mathbb{P}^r$ which projects onto the n -fold scroll X . We get a set-theoretical description of X as a degeneracy locus, in the same manner as in [10]. We can no longer conclude as in [10], for the ideal sheaf of X and of the degeneracy locus no longer differ along a 0-dimensional scheme, but along $d - r + n$ fibers of X . We will see that this degeneracy locus is fibered over C and obtained from X by "replacing" $d - r + n$ fibers of X by multiple fibers and this extra information will be enough to bypass this problem.

Proof. The k -regularity of X is equivalent to the $k - 1$ -normality of X and the vanishing of $H^i(\mathcal{O}_X(k - 1 - i))$. Note that by degeneration of the Leray spectral sequence

$$H^i(X, \mathcal{O}_X(d - r + n - i)) \simeq H^i(C, \pi_*(\mathcal{O}_X(d - r + n - i))),$$

so that we only have to show that

$$H^i(\mathbb{P}^r, \mathcal{I}_{X|\mathbb{P}^r}(d - r + n + 1 - i)) = 0 \text{ for } i = 1, 2.$$

The embedding $\mathbb{P}(E) \xrightarrow{p} X \subset \mathbb{P}^r = \mathbb{P}(V)$ factors through an embedding h of $\mathbb{P}(E)$ in $C \times \mathbb{P}^r$ and the second projection f onto \mathbb{P}^r , hence corresponds to the data of an exact sequence of \mathcal{O}_C -modules

$$0 \rightarrow M \xrightarrow{i_p} V \otimes_{\mathbb{C}} \mathcal{O}_C \rightarrow E \rightarrow 0$$

such that $H^1(C, M) = 0$. Let us denote by π the first projection from the product $C \times \mathbb{P}^r$. From the Euler exact sequence on \mathbb{P}^r , twisted by $\mathcal{O}_{\mathbb{P}^r}(1)$ and pulled back by f to $C \times \mathbb{P}^r$ and from the pull back by π of the morphism i_p we get a morphism $\pi^*M \xrightarrow{s} f^*(\mathcal{O}_{\mathbb{P}^r}(1))$ of $\mathcal{O}_{C \times \mathbb{P}^r}$ -modules which vanishing locus is schematically $h(\mathbb{P}(E))$.

The Koszul complex associated to $\pi^*(M) \otimes f^*(\mathcal{O}_{\mathbb{P}^r}(-1)) \xrightarrow{s \otimes f^*(\mathcal{O}_{\mathbb{P}^r}(-1))} \mathcal{O}_{C \times \mathbb{P}^r}$ gives a locally free resolution of $h_*(\mathcal{O}_{\mathbb{P}(E)})$:

$$0 \rightarrow \pi^*(\wedge^{r-n+1} M) \otimes f^*(\mathcal{O}_{\mathbb{P}^r}(n - r - 1)) \rightarrow \cdots \rightarrow \pi^*(\wedge^j M) \otimes f^*(\mathcal{O}_{\mathbb{P}^r}(-j)) \rightarrow \cdots \rightarrow \pi^*(\wedge^2 M) \otimes f^*(\mathcal{O}_{\mathbb{P}^r}(-2)) \rightarrow \pi^*(M) \otimes f^*(\mathcal{O}_{\mathbb{P}^r}(-1)) \rightarrow \mathcal{O}_{\mathbb{P}^r} \rightarrow h_*(\mathcal{O}_{\mathbb{P}(E)}) \rightarrow 0.$$

Indeed this complex is exact since $s \otimes f^*(\mathcal{O}_{\mathbb{P}^r}(-1))$ is generically surjective and its cokernel $h_*(\mathcal{O}_{\mathbb{P}(E)})$ is of expected codimension $(rk(M) - 1 = r - n)$ in the locally Cohen-Macaulay scheme $\mathcal{O}_{C \times \mathbb{P}^r}$ ([17] p. 344).

Since the article of Gruson, Lazarsfeld and Peskine, it is now a standard trick , to twist this resolution by the pull back by π of a suitable line bundle A on C , in order to be able to push down to X in \mathbb{P}^r the pieces of information of this resolution of $h(\mathbb{P}(E))$.

Lemma 3.1. *There is a line bundle A on C which satisfies*

$$h^1(M \otimes A) = 0, \quad h^1(\wedge^2 M \otimes A) = 0, \quad h^1(A) = 0.$$

For such a line bundle we have then $h^0(M \otimes A) = (r - n)(d - r + n) + 1$ and $h^0(A) = d - r + n$.

Proof. The proof of this is similar to [10] (lemma 1.7). First we have a strictly decreasing filtration by vector bundles F_i such that $\frac{F_i}{F_{i+1}} = L_i$. is a line bundle of negative degree

$$M = F_1 \supset F_2 \supset \cdots \supset F_{r+1-n} \supset 0 = F_{r+2-n}$$

In order to get $H^1(C, M \otimes A) = 0$ and $H^1(C, \wedge^2 M \otimes A) = 0$, it is enough to choose A such that $H^1(C, L_i \otimes A) = 0$ and $H^1(C, L_i \otimes L_j \otimes A) = 0$.

Since a generic line bundle of degree $\geq g-1$ is non special, it is enough to take A generic, so that $\deg(A) + r - n - 1 - d \geq g-1$, that is to say $\deg(A) \geq g - r + n + d = d - \text{codim}(h(\mathcal{O}_{\mathbb{P}(E)})) + g \geq g-1$. Indeed we have $\deg(L_i \otimes L_j) \geq -d + r - n - 1$.

Since $rk(M) = r + 1 - n$ the smallest possible degree for $\deg(L_i)$ is $r - n - d$, so that we also have $H^1(C, M \otimes A) = 0$. Note that $h^1(C, A) = 0$, so that by Riemann-Roch's theorem $h^0(C, A) = d + n - r + 1$. Since $h^1(C, M \otimes A) = 0$ and $h^1(C, L_i \otimes A) = 0$ we have $h^0(C, M \otimes A) = (r - n)(d + n - r) + 1$. \square

Applying Künneth's formula to the push forward by f of the Koszul resolution of $h_*(\mathcal{O}_{\mathbb{P}(E)})$ twisted by $\pi^*(A)$ we get the following complex

$$\begin{aligned} 0 &\rightarrow H^0(\wedge^{r-n+1} M \otimes A) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^r}(n - r - 1) \rightarrow \cdots \rightarrow H^0(\wedge^2 M \otimes A) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^r}(-2) \\ &\rightarrow H^0(M \otimes A) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^r}(-1) \xrightarrow{u} H^0(A) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^r} \rightarrow p_*(h^*\mathcal{O}_{\mathbb{P}(E)} \otimes \pi^*(A)) \rightarrow 0. \end{aligned}$$

Let t denote the C -projective bundle structure map of $\mathbb{P}(E)$. For the choice of line bundle A as in the previous lemma, by the same argument as in ([10], (1.5)), this complex is exact. We get $f_* \circ h_*(t^*A) \simeq f_*(h_*(\mathcal{O}_{\mathbb{P}(E)}) \otimes \pi^*A)$ by the projection formula.

The degeneracy locus of the morphism $\mathcal{O}_{\mathbb{P}^r}^{(r-n)(d+n-r)+1}(-1) \xrightarrow{u} \mathcal{O}_{\mathbb{P}^r}^{d+n-r+1}$ that we obtain this way is set-theoretically $X = p(h(\mathcal{O}_{\mathbb{P}(E)}))$. The cokernel of u is isomorphic to p_*t^*A by the projection formula applied to $p = f \circ h$ and $t = \pi \circ h$.

A generic section σ of A induces an exact sequence

$$0 \rightarrow \mathcal{O}_C \xrightarrow{\sigma} A \rightarrow A \otimes \mathcal{O}_D \rightarrow 0$$

whith $D = \sum_{i=1}^{d+n-r} p_i$ and where the p_i 's are distinct points of C . The push forward by p of this exact sequence is

$$0 \longrightarrow p_*\mathcal{O}_{\mathbb{P}(E)} \longrightarrow p_*t^*A \longrightarrow \mathcal{F} \longrightarrow R^1p_*\mathcal{O}_{\mathbb{P}(E)} = 0,$$

where \mathcal{F} denotes $p_*(t^*(A \otimes \mathcal{O}_D))$.

Moreover since p is a birational morphism and X is smooth, $p_*(\mathcal{O}_{\mathbb{P}(E)}) \simeq \mathcal{O}_X$. On each fiber $t^{-1}(p_i)$, we have $t^*(A)_{p_i} \simeq \mathcal{O}_{\mathbb{P}(E), p_i}$. Therefore we get $t^*(A \otimes \mathcal{O}_D) \simeq t^*\mathcal{O}_D$, so that $\mathcal{F} \simeq \bigoplus_{i=1}^{d-r+n} \mathcal{O}_{X_i}$ where X_i is the fiber of the scroll X over the point p_i .

As in [10] (proof of theorem 2.1 p. 500) the existence of the commutative diagram

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
& & \langle t^* \sigma \rangle \otimes \mathcal{O}_{\mathbb{P}^r} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\
& & \downarrow & & \downarrow p_* t^* \sigma & & \\
0 & \longrightarrow & K & \longrightarrow & \mathcal{O}_{\mathbb{P}^r}^l(-1) & \xrightarrow{u} & H^0(t^* A) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^r} \longrightarrow p_* t^* A \longrightarrow 0 \\
& & \downarrow & & \parallel & \downarrow & \downarrow \\
0 & \longrightarrow & N & \longrightarrow & \mathcal{O}_{\mathbb{P}^r}^l(-1) & \xrightarrow{v} & \mathcal{O}_{\mathbb{P}^r}^{d+n-r} \longrightarrow \mathcal{F} \longrightarrow 0 \\
& & \downarrow & & \downarrow & \downarrow & \downarrow \\
& & \mathcal{I}_{X|\mathbb{P}^r} & & 0 & & 0 \\
& & \downarrow & & & & \\
& & 0 & & & &
\end{array},$$

which allows us to compare the regularity of X with the regularity of N , follows from the minimality of

$$0 \rightarrow K = \ker(u) \rightarrow \mathcal{O}_{\mathbb{P}^r}^l(-1) \xrightarrow{u} H^0(t^* A) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^r} \rightarrow p_* t^* A.$$

In [10] the morphism corresponding to u is constructed from a minimal free resolution; the local minimality we seek here is taken care of by the following lemma.

Lemma 3.2. (*A minimality criterion*) Let L be the Koszul complex associated to some map $L_1 \xrightarrow{s} R$ where R is a local ring of maximal ideal \mathfrak{n} . Let us assume that there is a local morphism $S \hookrightarrow R$ turning R into a finite free S -module of rank l . Any R -module N has via this morphism an S -module structure that we denote by $s|N$. Consider the map of S -modules $s|L_1 \xrightarrow{\phi} S$, obtained by composing $s|L_1 \xrightarrow{s|s} R$ with $R \xrightarrow{b \wedge -} S \simeq \wedge^l R$, where b is the wedge product of the first $h-1$ elements of a base of the free S -module $s|L_1$. The Koszul complex K of S -modules associated to ϕ surjects onto the complex of free modules $s|L$. In particular this last complex is minimal if K is minimal. Moreover if $s|s(L_1) \subset \mathfrak{m}R$, where \mathfrak{m} is the maximal ideal of S , then the complex L is minimal.

Proof. The natural surjection of R -modules

$$\begin{aligned}
R \otimes_S s|L_1 &\xrightarrow{\psi} L_1 \\
x \otimes l &\longmapsto xl
\end{aligned}$$

gives a commutative diagram

$$\begin{array}{ccc}
L_1 & \xrightarrow{s} & R \\
\psi \uparrow & & \uparrow \wr \\
R \otimes_S s|L_1 & \xrightarrow{R \otimes_S \phi} & R
\end{array}$$

that induces a surjective map of complexes of free R -modules between $K \otimes_S R$ and L . Indeed by construction the map of S -modules $b \wedge -$ is surjective hence

$(b \wedge -) \otimes_S R$ is also surjective. It thus induces an automorphism σ of R . It is then clear that $s \circ \psi = \sigma^{-1} \circ R \otimes_S \phi$. The surjection between the complexes of R -modules induced by ψ gives a surjection $K \rightarrow {}_S|L$, as claimed.

If ${}_S|s({}_S|L_1) \subset \mathfrak{m}R$ after wedging by b , we have $\phi({}_S|L_1) \subset \mathfrak{m}S$ so that ${}_S|L_1 \xrightarrow{\phi} S$ is minimal when ${}_S|L_1 \xrightarrow{s|s} {}_S|R$ is minimal. \square

By Künneth formula $f_*(\pi^*(A)) = H^0(A) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^r}$, so that $f_*\pi^*A$ is a finite locally free $\mathcal{O}_{\mathbb{P}^r}$ -module of rank $h^0(A)$. We are exactly in the situation of the criterion since locally, the complex $K.(s) \otimes \pi^*A$ is simply the Koszul complex associated to $\pi^*(M)_x \otimes \pi^*A_x \rightarrow \pi^*A_x$. Since the map $H^0(A) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^r} \rightarrow p_*t^*A$ induces a surjection on global sections, none of the sections defining $H^0(A) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^r} \rightarrow p_*t^*A$ vanishes. This shows then that $0 \rightarrow K \rightarrow \mathcal{O}_{\mathbb{P}^r}^l(-1) \xrightarrow{u} H^0(t^*A) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^r} \rightarrow p_*t^*A$ is locally minimal.

Lemma 3.3. *We have $H^2(K(k)) = 0$ for all $k > -n$.*

Proof. This is equivalent to show that $H^1(C \times \mathbb{P}^r, \mathcal{I}_{\mathbb{P}(E)|\mathbb{P}^r} \otimes \pi^*A \otimes f^*\mathcal{O}_{\mathbb{P}^r}(k)) = 0$ for all $k > -n$. Indeed since f is projective and only has fibers of dimension 1, $R^j f_*$ vanishes on any coherent sheaf for $j \geq 2$, so that

$$R^1 f_*(\pi^*(M \otimes A) \otimes f^*(\mathcal{O}_{\mathbb{P}^r}(k-1))) \rightarrow R^1 f_*(\mathcal{I}_{\mathbb{P}(E)|\mathbb{P}^r} \otimes \pi^*A \otimes f^*\mathcal{O}_{\mathbb{P}^r}(k))$$

is surjective. By non speciality of A and use of the Künneth formula we get $R^1 f_*(\mathcal{I}_{\mathbb{P}(E)|\mathbb{P}^r} \otimes \pi^*A \otimes f^*\mathcal{O}_{\mathbb{P}^r}(k)) = 0$. \square

Consider now the Koszul resolution of $h_*(\mathcal{O}_{\mathbb{P}(E)}$ twisted by $f^*(\mathcal{O}_{\mathbb{P}^r}(k))$

$$\cdots \rightarrow \pi^*(\wedge^2 M \otimes A) \otimes f^*(\mathcal{O}_{\mathbb{P}^r}(k-2)) \rightarrow \pi^*(M \otimes A) \otimes f^*(\mathcal{O}_{\mathbb{P}^r}(k-1)) \rightarrow \\ \pi^*A \otimes f^*(\mathcal{O}_{\mathbb{P}^r}(k)) \rightarrow h_*\mathcal{O}_{\mathbb{P}(E)} \otimes f^*(\mathcal{O}_{\mathbb{P}^r}(k)) \rightarrow 0.$$

Let us denote by \mathcal{F}_j the cokernel of

$$\pi^*(\wedge^{j+1} M \otimes A) \otimes f^*(\mathcal{O}_{\mathbb{P}^r}(k-j-1)) \longrightarrow \pi^*(\wedge^j M \otimes A) \otimes f^*(\mathcal{O}_{\mathbb{P}^r}(k-j)).$$

Lemma 3.4. *We have $R^i \pi_*(\mathcal{I}_{\mathbb{P}(E)|\mathbb{P}^r} \otimes \pi^*A \otimes f^*\mathcal{O}_{\mathbb{P}^r}(k)) = 0$ for $i \geq 1$.*

Proof. Since the fibers of π are of dimension r and π is projective we know that $R^{r+1}\pi_*$ vanishes on any coherent sheaf, so that $R^{r+1}\pi_*$ is right exact. Then $R^r \pi_*(\pi^*(\wedge^j M \otimes A) \otimes f^*\mathcal{O}_{\mathbb{P}^r}(k-j)) = \wedge^j M \otimes A \otimes H^r(\mathcal{O}_{\mathbb{P}^r}, \mathcal{O}_{\mathbb{P}^r}(k-j)) = 0$ for all $k > -n$, since $j \leq rk(M) = r + 1 - n$ and $k - j > -r - 1$. Therefore $R^r \mathcal{F}_j = 0$ for all j . Repeating this argument, we deduce that $R^i \pi_* \mathcal{F}_j = 0$ for all j when $k > -n$ and $H^1(C, \pi_*(\pi^*(M \otimes A) \otimes f^*\mathcal{O}_{\mathbb{P}^r}(k))) = H^1(C, M \otimes A) \otimes H^0(\mathcal{O}_{\mathbb{P}^r}, \mathcal{O}_{\mathbb{P}^r}(k)) = 0$. Moreover $H^2(C, \pi_* \mathcal{F}_1) = 0$, since $\pi_* \mathcal{F}_1$ is coherent (π is projective) and $\dim C = 1$. Therefore $H^1(\pi_* \mathcal{F}_0) = H^1(\pi_*(\mathcal{I}_{\mathbb{P}(E)|\mathbb{P}^r} \otimes \pi^*A \otimes f^*\mathcal{O}_{\mathbb{P}^r}(k))) = 0$. \square

To show that X is $d - r + n + 1$ -regular, it suffices to show that $H^i(N(k)) = 0$ for $k = d - r + n + 1 - i$ and $i = 1, 2$.

Lemma 3.5. *We have $H^i(N(k)) = 0$, for $k = d - r + n + 1 - i$ and $i = 1, 2$.*

We deduce from the local minimality of

$$0 \rightarrow K = \ker(u) \rightarrow \mathcal{O}_{\mathbb{P}^r}^l(-1) \xrightarrow{u} H^0(t^*A) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^r} \rightarrow p_*t^*A$$

that the following complex extracted from the previous commutative diagram

$$0 \longrightarrow N \longrightarrow \mathcal{O}_{\mathbb{P}^r}^l(-1) \xrightarrow{v} \mathcal{O}_{\mathbb{P}^r}^{d+n-r} \longrightarrow 0 \quad (*)$$

is also locally minimal.

Now the sheafification of a minimal resolution of $\bigoplus_{l \in \mathbb{Z}} H^0(\mathcal{F}(l))$ is of the form

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^r}^{n_r-1}(-r-1) \rightarrow \cdots \rightarrow \mathcal{O}_{\mathbb{P}^r}^{(r+1-n)(d+n-r)}(-1) \xrightarrow{w} \mathcal{O}_{\mathbb{P}^r}^{d-r+n} \rightarrow \mathcal{F} \rightarrow 0.$$

Since the exact sequence $(*)$ comes from the sheafification of a locally free minimal resolution of $\bigoplus_{l \geq l_0} H^0(\mathcal{F}(l))$ for some $l_0 \geq 0$, we can construct the following commutative diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N & \longrightarrow & \mathcal{O}_{\mathbb{P}^r}^l(-1) & \xrightarrow{v} & Im(v) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \iota \\ 0 & \longrightarrow & P & \longrightarrow & \mathcal{O}_{\mathbb{P}^r}^{(r+1-n)(d-r+n)}(-1) & \xrightarrow{w} & Imw \longrightarrow 0 \\ & & \phi \downarrow & & \downarrow & & \\ & & \mathcal{O}_{\mathbb{P}^r}^{d+n-r-1}(-1) & \xrightarrow{\cong} & \mathcal{O}_{\mathbb{P}^r}^{d+n-r-1}(-1) & & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

Since P has a resolution of type

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^r}^{n_r+1}(-r-1) \longrightarrow \cdots \longrightarrow \mathcal{O}_{\mathbb{P}^r}^{n_2}(-2) \xrightarrow{w_1} P \longrightarrow 0,$$

we have $H^i(\mathbb{P}^r, P(d-r+n+1-i)) = 0$ for $i = 1, 2$ (by lemma 1.6 of [10]).

Therefore to show that $H^i(\mathbb{P}^r, N(d-r+n+1-i)) = 0$ for $i = 1, 2$, it suffices to show that $\ker(q)$ is $d-r+n+1$ -regular, where $\ker(q)$ is defined by the following triangular diagram

$$\begin{array}{ccccc} \mathcal{O}_{\mathbb{P}^r}^{n_2}(-2) & \xrightarrow{w_1} & P & \longrightarrow & 0 \\ & \searrow q & \downarrow \phi & & \\ & & \mathcal{O}_{\mathbb{P}^r}^{d+n-r-1}(-1) & & \\ & & \downarrow & & \\ & & 0 & & \end{array}$$

This holds by the now standard trick due to Gruson, Lazarsfeld and Peskine, consisting in applying Lemma 1.6 of [10] to the Eagon-Northcott complex associated to q twisted by $\mathcal{O}_{\mathbb{P}^r}(d+n-r-1)$. \square

REFERENCES

- [1] A. Alzati, *A new Castelnuovo bound for two codimensional subvarieties in \mathbb{P}^r* , Proc. Amer. J. Soc., 114, (1992), p. 607-611.
- [2] D. Bayer and D. Mumford, *What can be computed in algebraic geometry?*, Computational algebraic geometry and commutative algebra (Cortona 1991), p. 1-48, Sympos. Math. XXXIV, Cambridge Univ. Press, Cambridge (1993).
- [3] A. Beauville, Surfaces algébriques complexes, Astérisque, vol. 54, Société Mathématique de France, (1978).

- [4] M. A. Bertin, *On the regularity of varieties having an extremal secant line*, Ph.D. thesis, Columbia University, (2000).
- [5] E. Bertini, Introduzione alla geometria proiettiva degli iperspazi, Enrico Spoeri, Pisa , (1907).
- [6] G. Castelnuovo, *Sui multipli di una serie lineare di gruppi di punti appartenente ad una curva algebrica*, rend. Circ. Mat. Palermo, **7** (1893).
- [7] _____, Memorie Scelte, Nicola Zanichelli, Bologna 1937
- [8] D. Eisenbud and S. Goto, *Linear free resolutions and minimal multiplicity*, J.Alg., **88**, (1984) p. 89-133
- [9] D.Eisenbud and J.Harris, *Varieties of minimal degree (a centennial account)*, Bowdoin conference in Algebraic Geometry (Brunswick, Maine 1985), Proceedings of Symposia in Pure Mathematics XXXVI, part 1, p. 1-13, Am. Math. Soc., Providence RI (1987).
- [10] L. Gruson, R. Lazarsfeld, Ch. Peskine, *On a theorem of Castelnuovo, and the equations defining space curves*, Inv. Math , **72**, (1983)
- [11] R. Lazarsfeld, *A Castelnuovo bound for smooth surfaces*, Duke J. 1987
- [12] R. Lazarsfeld, *Private notes, June 1982, communicated in May 1999.*
- [13] M. Maruyama, *Elementary transformations in the theory of algebraic vector bundles*, Algebraic Geometry (La Rabida 1981), (1982) p. 241-266
- [14] D. Mumford, Lectures on curves on an algebraic surface, Annals of Math. studies, vol. 59, Princeton Univ. Press, (1966)
- [15] C.Okonek, M.Schneider and H.Spindler, Vector Bundles on Complex Projective Spaces, Progress in Math , vol. 3 , Birkhäuser, (1980)
- [16] H.C.Pinkham, *Castelnuovo bound for smooth surfaces*, Invent.Math., (1986) p. 321-332
- [17] L. Szpiro, *Sur les travaux de Kempf, Kleinman et Laskov sur les diviseurs exceptionnels*, Séminaire Bourbaki, exposé no. 417, Lecture Notes in Math, **317**, Springer V.

E-mail address: mabertin@fourier.ujf-grenoble.fr

Current address: Université de Grenoble I, Institut Fourier, 38402 St Martin-d'Hères, FRANCE